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PROPERTIES OF PROPOSITIONAL METRIC TEMPORAL CALCULUS FOR DESCRIPTION OF EVOLVING CONCEPTUALIZATION
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Abstract
For a propositional metric temporal calculus *PTC(MT)* based on the known language of a propositional metric temporal logic introduced by Arthur Prior defined are negation normal form (n.n.f.) and *FnPn*-normal form (*FnPn*-n.f.) of a *PTC(MT)* formula. Proved is existence of n.n.f. and *FnPn*-n.f. for arbitrary *PTC(MT)* formulae. Beth-Kripke semantic tableaux method is used to prove completeness and soundness of *PTC(MT)*.

1. Introduction
Knowledge engineering is one of the fields of artificial intelligence, focusing in knowledge mining and formalization [1, pp.59-61]. One of the approaches to present a domain of interest - ontological engineering – considers conceptual structures (also called *intensional* conceptualizations) of the domain and presents them in ontology. According to Guarino [2] "ontology is a logical theory accounting for the intended meaning of a formal vocabulary". Ontologies as formal theories are often expressed in some logical language, varying from language of first-order predicate logic (e.g. Knowledge Interchange Format – KIF) to the family of Description Logics (dealing with unary and binary predicates only).

Most of the languages used for ontology description are static, i.e. they consider an intensional conceptualization of an arbitrary domain in a static manner. However the real world is dynamic and evolution may influence conceptualization of the domain as well.

One of the ways to incorporate the dynamic nature of the domain into an information system focuses on the usage of an aletic modal logic for declaration of "allowed" states of affairs within this domain [3]. According to the approach of [3] an *extensional* conceptualization (which in terms of [2] consists of domain concepts instances/relationships) is allowed to change only as prescribed by the "dynamics" axioms of a formal theory based on the aletic modal logic language.

However in [3, p.154] it is outlined that of great importance is the dynamic nature of the *intensional* conceptualization itself.

Incorporating such evolving intensional conceptualizations into information systems one may require to analyze changes occurring in that conceptualization at a certain point of time, to trace a history of a concept/relationship within the sequence of conceptualizations of the same domain, to discover logical equivalence/subsumption of ontologies describing intensional conceptualizations of the same domain at different time points.

Extension of a logical language for the ontology description with an explicit notion of time is the way to provide the aforementioned analysis in the declarative way.

Temporal logic is a kind of symbolic modal logic [4] dealing with domain description statements, which are interpreted over the time flow, either point-based or interval-based.

The reviews of known logical systems involving temporal modalities can be found in [5], [6], [7]. Among these systems are Lemmon's minimal system K_t (with the unary operators *F* – "somewhere in the future", *G* – "always in the future" and their mirrors), von Wright system "And then" (the binary operator T_w, and basic construct $pT_w q$ – "*p* and then *q*"), Scott's system "And next instant" (the unary operator T_s, basic construct $T_s p$ – "in the next time point will be *p*").

Of special interest is the metric temporal logic [4] with modalities *Fn* ("it will be the case after *n* time points") and *Pn* ("it was the case *n* time points ago"), which allow to explicitly state at what time point an event occurs.

2. Problem formulation

Let's consider an example of a correct formula of the metric temporal logic. Formula $Fn(p \supset q)$, where *p* is a propositional variable for the sentence "to be child", and *q* is a propositional variable for the sentence "to be human" means that in *n* time points (after the current time point) it will be true that being a child implies being a human.

Reasoning tasks over such formulae includes satisfiability checking of arbitrary formula of the metric logic, or, in other terms, whether this formula has a model. Back to example, one may check satisfiability of the formula *Fnq* or $Fn(p \wedge q)$.

The aim of the presented research is to construct the correspondent calculus, its interpretation and the inference mechanism to check existence of the model for arbitrary formulae.

The paper presents propositional metric temporal calculus *PTC(MT)* based on the known language of propositional metric temporal logic *LMP* [4]. Main properties of this calculus are analyzed, namely *completeness* (whether all logically valid formulae of a formal theory are theorems of that theory) and *soundness* (which means that the formal theory does not allow to prove both a theorem and its contradiction).

For these purposes we need to adopt a technique called *tableaux* (namely, Beth's semantic tableaux method [8]), which is usual for proving completeness and soundness of a modal logic [8]. The survey of tableau systems for modal and temporal logics may be found in [9]. Adaptation of the method for von Wright system "And then" (including its quantified extension) was done by Solodukhin in [10].

Tableau rules usually operate on a formula containing negation only for atomic subformulae. Thus we need to define negation normal form (n.n.f.) for *PTC(MT)*.

Introduction of the temporal modalities *Fn* and *Pn* will influence the initial set of tableau rules to deal with formulae containing these modalities. We will introduce *FnPn*-normal form (*FnPn*-n.f.) of a *PTC(MT)* formula and prove the existence of n.n.f. and *FnPn*-n.f. for arbitrary *PTC(MT)* formulae. The set of time points will be considered as the set of integers.

3. Results

Propositional metric temporal calculus considers time having linear discrete structure, infinite into the past and to the future, assumes that time points are organized with reflexive and transitive ordering relation.

Such structure of time is isomorphic to the structure $\langle Z, < \rangle$, where Z - is a set of integers, and < - is a strict ordering relation.

3.1 Alphabet, formulae, axioms, deduction rules

The alphabet of *PTC(MT)* consists of:

(a) Propositional variables $p, q, r, s, ...$;

(b) Primitive propositional connectives $\neg, \supset$, and additional connectives $\wedge, \vee, \equiv$, defined over primitive ones in the usual way;

(c) Temporal operators *Fn*, *Pn* (*Fn* – « it will be the case after *n* time points», *Pn* – «it was the case *n* time points ago»);

PTC(MT) terms are:

(a) $\nu, \nu_1, \nu_2, ...$ are natural numbers and «0»;

(b) $i, i_1, ..., j, j_1, ...$ are numerical variables;

(c) if $n_1, ..., n_m$ are natural numbers and «0» or numerical variables, and θ– *m*-ary operator, then $\theta(n_1, ..., n_m)$ – is a term.

Formulae are constructed following the rules:

(a) Every propositional variable is a formula;

(b) If φ and ψ are formulae, then $\neg\varphi$, $\varphi \supset \psi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \equiv \psi$ are also formulae;

(c) If Pr^m is a predicate letter denoting *m*-ary predicate, defined over integers (e.g., «=», «>»,…), and $n_1, ..., n_m$ – are terms, then $\mathrm{Pr}^m(n_1, ..., n_m)$ - is a formula;

(d) If φ – is a formula, then $Fn\varphi, Pn\varphi$, $\exists i\varphi, \forall i\varphi$ – are also formulae.

Alphabet of *PTC(MT)* is defined.

Definition 1.

Numerical variable *i* **occurs free** in a formula φ, if it is not within the scope of any quantifier in φ.

Definition 2.

Term *n* is **free in a formula φ for a numerical variable** *j*, if there are no free occurrences of *j* in φ, such that *j* is within the scope of any quantifier $\forall i_m$, where i_m is a numerical variable in the term *n*.

PTC(MT) axioms set will consist of all axioms of the propositional calculus and some axioms of temporal logic, taken from [4],[6],[7].

Following formulae are axioms (propositional axioms are correspondent to L_4 system, see [11, p.49]):

(A1) $p \supset (q \supset p)$;

(A2) $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$;

(A3) $p \wedge q \supset p$

(A4) $p \wedge q \supset q$

(A5) $p \supset (p \vee q)$

(A6) $q \supset (p \vee q)$

(A7) $p \supset (q \supset (p \wedge q))$

(A8) $(p \supset q) \supset ((r \supset q) \supset ((p \vee r) \supset q)$

(A9) $(p \supset q) \supset ((p \supset \neg q) \supset \neg p)$

(A10) $\neg\neg p \supset p$

(AMT1) $(\neg Fn \neg (p \supset q)) \supset (Fnp \supset Fnq)$ – logical homogeneity in the future

(AMT1.1) $(\neg Pn \neg (p \supset q)) \supset (Pnp \supset Pnq)$ – logical homogeneity in the past

(AMT2) $Fn \neg Pn \neg p \supset p$

(AMT2.1) $Pn \neg Fn \neg p \supset p$

(AMT3) $Fm \exists iFip \supset \exists iFmFip$

(AMT3.1) $Pm \exists iPip \supset \exists iPmPip$

(AMT4) $Fm \exists iPip \supset \exists iFmPip$

(AMT4.1) $Pm \exists iFip \supset \exists iPmFip$

(AMT5) $F(m+n)p \supset FmFnp$

(AMT5.1) $P(m+n)p \supset PmPnp$

(AMT6) $\neg Fnp \supset Fn \neg p$ – infinity into the future

(AMT6.1) $\neg Pnp \supset Pn \neg p$ – infinity into the past

(AMT7) $Fn \neg p \supset \neg Fnp$ – nonbranching in the future

(AMT7.1) $Pn \neg p \supset \neg Pnp$ – nonbranching in the past

(AMT8) $FmFnp \supset F(m+n)p$ – transitivity in the future

(AMT8.1) $PmPnp \supset P(m+n)p$ – transitivity in the past

(AMT9) $(m = n + k) \supset (FmPnp \supset Fkp)$ – iteration of temporal modalities

Propositional axioms are independent with respect to *PTC(MT)*, the same applies for temporal axioms.

Deduction rules for calculus *PTC(MT)* are:

(R1) $\dfrac{\varphi, \varphi \supset \psi}{\psi}$ – Modus Ponens

(R2) $\dfrac{\varphi(p)}{\psi(p/\gamma)}$ – substitution rule (ψ is obtained after replacing in φ all occurrences of a propositional variable p with formula γ)

(R3) $\dfrac{\varphi}{\neg Fn\neg\varphi}$ – the rule of deriving "always in the future"

(R4) $\dfrac{\varphi}{\neg Pn\neg\varphi}$ – the rule of deriving "always in the past"

Let φ be a *PTC(MT)* formula that does not contain numerical variable *i*, $\varphi[j/i]$ be a *PTC(MT)* formula with all free occurrences of a numerical variable *j* replaced with *i*. Then the following deduction rule may be applied:

(R5) $\dfrac{\varphi[j/i]}{\forall i\varphi}$ – the generalization rule

If φ is a *PTC(MT)* formula which contains numerical variable *i*, and $\varphi[i/n]$ be a *PTC(MT)* formula with all occurrences of a numerical variable *i* replaced with term *n*, which is free for *i* in φ, then the following deduction rule may be applied:

(R6) $\dfrac{\forall i\varphi}{\varphi[i/n]}$

Calculus is constructed.

Throughout this paper we restrict the discussion with binary operations "+", "–" for *PTC(MT)* terms construction and use the only binary predicate "="("equality").

3.2 Investigation of the properties of *PTC(MT)*.

There are several theorems provable in the *PTC(MT)*, which will facilitate further analysis of completeness and soundness of this calculus. All the theorems T1-T25 are proved by the rule of contraries.

Theorem 1.

(**T1**) $\vdash Fn(p\wedge q)\equiv Fnp\wedge Fnq$

Proof: necessity is proved with the help of AMT6, the consequence of the deduction theorem (see [11, p.40, consequence 1.9 (i)], the deduction theorem. Sufficiency is proved using the definition of $\vee,\wedge,\supset$, A4, AMT1, AMT7.

Theorem 2.

(**T2**) $\vdash Pn(p\wedge q)\equiv Pnp\wedge Pnq$

Theorem 3.

(**T3**) $\vdash Fn(p\vee q)\equiv Fnp\vee Fnq$

Proof: necessity is proved with the help of the definition of $\vee$, A10, AMT6, AMT7, AMT1. Sufficiency is proved, basing on the definition of $\vee$, distributive law for $\wedge,\vee$, A10, AMT6, AMT7, T1.

Theorem 4.

(**T4**) $\vdash Pn(p \vee q) \equiv Pnp \vee Pnq$

Theorem 5.

(**T5**) $\vdash (m = n + k) \supset (FmPnp \equiv Fkp)$

Proof: $(m = n + k) \supset (FmPnp \supset Fkp)$ is the axiom AMT9. Sufficiency ($\vdash (m = n + k) \supset (Fkp \supset FmPnp)$) is proved with the help of the definition of $\vee$, A10, AMT6, AMT7, AMT5, AMT2.

Theorem 6.

(**T6**) $\vdash (n = k + m) \supset (FmPnp \equiv Pkp)$

Proof: necessity can be proved using the definition of $\vee$, AMT5.1, A10, AMT6, AMT7, AMT7.1, AMT2.1. Sufficiency is proved using the definition of $\vee$, AMT6, AMT7, AMT5.1, A10, AMT6, AMT7.

Theorem 7.

(**T7**) $\vdash (m = k + n) \supset (PmFnp \equiv Pkp)$

Theorem 8.

(**T8**) $\vdash (n = k + m) \supset (PmFnp \equiv Fkp)$

Proof of the theorems T7 and T8 is analogous to proof of T5 and T6.

Theorem 9.

(**T9**) $\vdash \forall iFip \supset \exists iFip$

Proof: use A10, AMT6, AMT7 and the definition of $\exists,\forall$.

Theorem 10.

(T10) $\vdash Fm\exists iFip \equiv \exists iFmFip$

Proof: $Fm\exists iFip \supset \exists iFmFip$ is the axiom AMT3. Sufficiency may be proved using the definition of $\vee$, T9, A4, AMT3, A3.

Similar proofs may be constructed for the following theorems, making the basis for grouping temporal modalities with quantifiers:

Theorems 11-13.

(T11) $\vdash Pm\exists iPip \equiv \exists iPmPip$

(T12) $\vdash Fm\exists iPip \equiv \exists iFmPip$

(T13) $\vdash Pm\exists iFip \equiv \exists iPmFip$

Theorems 14-17.

(T14) $\vdash Fm\forall iFip \equiv \forall iFmFip$

(T15) $\vdash Fm\forall iPip \equiv \forall iFmPip$

(T16) $\vdash Pm\forall iFip \equiv \forall iPmFip$

(T17) $\vdash Pm\forall iPip \equiv \forall iPmPip$

Theorems 18-21.

(T18) $\vdash \exists i \forall j FiFjp \equiv \exists iFi \forall jFjp$

(T19) $\vdash \exists i \forall j FiPjp \equiv \exists iFi \forall jPjp$

(T20) $\vdash \exists i \forall j PiFjp \equiv \exists iPi \forall jFjp$

(T21) $\vdash \exists i \forall j PiPjp \equiv \exists iPi \forall jPjp$

Theorems 22-25.

(T22) $\vdash \forall i \exists j FiFjp \equiv \forall iFi \exists jFjp$

(T23) $\vdash \forall i \exists j FiPjp \equiv \forall iFi \exists jPjp$

(T24) $\vdash \forall i \exists j PiFjp \equiv \forall iPi \exists jFjp$

(T25) $\vdash \forall i \exists j PiPjp \equiv \forall iPi \exists jPjp$

Definition 3.

Formula φ is called **atomic**, if φ is either a propositional variable or its negation, or a formula of the view $\Pr^m(n_1,...,n_m)$ or its negation.

Definition 4.

Formula φ is in **negation normal form** (n.n.f.), if for every subformula $\neg\psi$ formula ψ is atomic, and the whole formula φ is constructed without binary propositional connectives $\supset, \equiv$.

Theorem 26.

Let φ be a formula from *PTC(MT)*.

Then $\vdash \varphi \equiv \psi$, where ψ - is a formula in negation normal form (n.n.f.).

Theorem 26 may be proven by induction with help of axioms AMT1-AMT9 and theorems T1-T6.

Definition 5.

Formula φ is in ***FnPn*-normal form** (*FnPn*-n.f.), if it can be presented as:

$$\varphi \equiv \bigvee_{k=1}^{N} \Big(\bigwedge_{r^1=0}^{N_k^1} F\nu_{kr^1}\varphi_{kr^1} \wedge \bigwedge_{r^2=0}^{N_k^2} P\nu_{kr^2}\varphi_{kr^2} \wedge$$

$$\wedge \bigwedge_{r^3=0}^{N_k^3} \exists i_{kr^3} \Pr^{s+1}(i_{kr^3}, \nu_1, ...\nu_s) \wedge \bigwedge_{r^4=0}^{N_k^4} \forall i_{kr^4} \Pr^{s+1}(i_{kr^4}, \nu_1, ...\nu_s) \wedge$$

$$\wedge \bigwedge_{\substack{\alpha,\beta \\ r^j=\overline{0,N_k^j} \\ d=\overline{0,D_k^j}}} \alpha\, \nu_{kr^j} \beta\, i_{kr^j{}_1}\, \alpha\, i_{kr^j{}_1} ... \beta\, i_{kr^j{}_d}\, \alpha\, i_{kr^j{}_d}\, \varphi_{kr^j} \Big)$$

where :

- N - is a number of disjuncts in a formula,
- k - is an internal index for referencing disjuncts within the formula,
- $N_k^j \geq 0$ - is a number of conjuncts of a particular conjunct form within k-th disjunct

- $j = \overline{1,...,2^{2 \cdot D_k}}$ - is an index of a particular conjunct form within k-th disjunct,
- $\alpha \in \{F, P\}$ - is a symbol, partially denoting one of temporal modalities,
- $\beta \in \{\exists, \forall\}$ - is a symbol denoting one of quantifiers,
- $r^j = \overline{0,...,N_k^j}$ - is an internal index for referencing formulae of a particular conjunct form within k-th disjunct,
- $d = \overline{1,...,D_k^j}$ - is an internal index for referencing elements of the form $\beta_{kr^j}\ i_{kr^j}\ \alpha_{kr^j} i_{kr^j}$ within a formula in the r^j -th conjunct of the particular conjunct form within k-th disjunct,
- $D_k^j \leq D_k$ - is the number of quantifiers in the particular conjunct form within k-th disjunct,
- D_k - is the maximal number of quantifiers among all particular conjunct forms within k-th disjunct,
- φ_{kr^j} - are atomic formulae.

FnPn-n.f. of a *PTC(MT)* formula is a list of alternative histories of states of some object from a domain.

For example, the following formula is in *FnPn*-n.f.:

$\varphi = F1\varphi_1 \vee F1\varphi_2 \wedge F2\exists iFi\forall jFj\varphi_3$

Theorem 27.

Let φ be a formula of *PTC(MT)* in n.n.f. Then $\vdash \varphi \equiv \psi$, where ψ is a formula in *FnPn*-normal form.

Proof:

By theorem 26 every formula φ possesses its negation normal form. In what follows it will be considered that φ is already in n.n.f.

(By induction)

Step 1. If φ is an atomic formula, then it is already in *FnPn*-n.f.:

$\vdash \psi \equiv F0\varphi$.

If $\varphi \equiv p_1 \wedge p_2$, where p_1, p_2 are atomic, then $\vdash \psi \equiv F0(p_1 \wedge p_2)$ is in *FnPn*-n.f. If $\varphi \equiv p_1 \vee p_2$, then $\vdash \psi \equiv F0p_1 \vee F0p_2$ is in *FnPn*-n.f. Finally, formula of the view $\varphi \equiv Fnp$ or $\varphi \equiv Pnp$ is also in *FnPn*-n.f.

Step k. Let φ be a formula of *PTC(MT)* in *FnPn*-n.f.

Step (k+1).

Case 1. Let $\psi \equiv \neg\varphi$.

Using de Morgan laws, definition of $\exists, \forall$ and the axioms AMT6, AMT7, AMT6.1, AMT7.1 one can obtain ψ in *FnPn*-normal form.

Case 2. Let $\psi = \varphi_1 \vee \varphi_2$.

Can be proved basing on the properties of connectors $\wedge, \vee$.

Case 3. Let $\psi = \varphi_1 \wedge \varphi_2$

Can also be proved basing on the properties of connectors $\wedge, \vee$.

Case 4. Let $\psi \equiv Fn\varphi$.

Using axiom AMT8, theorems T1, T3, T5, T9 – T13 and rules R5, R6, prove that ψ in *FnPn*-n.f.

Case 5. Let $\psi \equiv Pn\varphi$.

Using axiom AMT8.1, theorems T2, T4, T6, T9 – T13 and rules R5, R6, prove that ψ in *FnPn*-n.f.

Case 6. Let $\psi \equiv \exists j \varphi$

Using theorems T9 – T25 and rules R5, R6 prove that ψ in *FnPn*-n.f.

Case 7. Let $\psi \equiv \forall j \varphi$

Using theorems T9 – T25 prove that ψ in *FnPn*-n.f.

Proof is completed.

3.3 Completeness and soundness of *PTC(MT)*.

Let's prove logical correctness of *PTC(MT)*.

Main properties of every formal theory are soundness and completeness with respect to the model. In modal logics models of formulae are constructed with the help of a set of rules, according to Beth's semantic tableaux technique.

Construct a model of an arbitrary *PTC(MT)* formula.

Definition 6.

Let φ be a formula in *FnPn*-n.f., and ψ be a subformula of φ. A sequence of formulae lists $<\zeta_0, \zeta_1, ..., \zeta_m, \zeta_{-1}, ...\zeta_{-s}>$, linearly ordered with a binary relation R (reflexive and transitive), forms a **chain** Z_φ for the formula φ, if this sequence is constructed following the set of rules, presented in the Table 1.

Table 1. Rules for construction of a semantic tableau for checking *PTC(MT)* formula satisfiability.

(0-rule)	Condition:	$\psi = \varphi$
	Action:	$\zeta_0 = \psi$
($\wedge$-rule)	Condition:	1. $\psi = \psi_1 \wedge \psi_2$ 2. $\{\psi_1, \psi_2\} \cap \zeta = \varnothing$
	Action:	$\zeta = \zeta \cup \{\psi_1, \psi_2\}$

($\vee$-rule)	Condition:	1. $\psi = \psi_1 \vee \psi_2$ 2. $\{\psi_1, \psi_2\} \cap \zeta = \varnothing$
	Action:	Either $\zeta = \zeta \cup \{\psi_1\}$ or $\zeta = \zeta \cup \{\psi_2\}$
($F\nu$-rule)	Condition:	1. $\psi = F\nu\psi_1$ 2. $\psi \in \zeta_k$ 3. $\nu \geq 1$
	Action:	1. If there is no $\zeta_{k+1} : \zeta_{k+1} \in \mathrm{Z}_\varphi$, then such list is created and new formula $\psi' = F(\nu-1)\psi_1$ is added to the ζ_{k+1}, $\psi' \in \zeta_{k+1}$ 2. If exists $\zeta_{k+1} : \zeta_{k+1} \in \mathrm{Z}_\varphi$, then $\psi' = F(\nu-1)\psi_1$ is added to the ζ_{k+1}, $\psi' \in \zeta_{k+1}$ 3. Between ζ_k and ζ_{k+1} relation $R(\zeta_k, \zeta_{k+1})$ is set.
($P\nu$-rule)	Condition:	1. $\psi = P\nu\psi_1$ 2. $\psi \in \zeta_k$ 3. $\nu \geq 1$
	Action:	1. If there is no $\zeta_{k-1} : \zeta_{k-1} \in \mathrm{Z}_\varphi$, then such list is created and new formula $\psi' = P(\nu-1)\psi_1$ is added to the ζ_{k-1}, $\psi' \in \zeta_{k-1}$ 2. If exists $\zeta_{k-1} : \zeta_{k-1} \in \mathrm{Z}_\varphi$, then $\psi' = P(\nu-1)\psi_1$ is added to the ζ_{k-1}, $\psi' \in \zeta_{k-1}$ 3. Between ζ_k and ζ_{k-1} relation $R(\zeta_k, \zeta_{k-1})$ is set.
$\exists iFi$-rule	Condition	1. $\psi = \exists iFi\psi_1$ 2. $\psi \in \zeta_k, \psi_1 \notin \zeta_k$
	Action	Either $\psi_1 \in \zeta_k$ or new formula $\psi' = F1\exists iFi\psi_1$ belongs to ζ_k, $\psi' \in \zeta_k$
$\exists iPi$-rule	Condition	1. $\psi = \exists iPi\psi_1$ 2. $\psi \in \zeta_k, \psi_1 \notin \zeta_k$

	Action	Either $\psi_1 \in \zeta_k$ or new formula $\psi' = P1\exists iPi\psi_1$ belongs to ζ_k, $\psi' \in \zeta_k$
$\forall iFi$-rule	Condition	1. $\psi = \forall iFi\psi_1$ 2. $\psi \in \zeta_k, \psi_1 \notin \zeta_k$
	Action	1. $\psi_1 \in \zeta_k$ 2. For each $\zeta_j : \zeta_j \in Z_\varphi, j > k$, such that the relation $R(\zeta_k, \zeta_j)$ is set, $\psi \in \zeta_j$
$\forall iPi$-rule	Condition	1. $\psi = \forall iPi\psi_1$ 2. $\psi \in \zeta_k, \psi_1 \notin \zeta_k$
	Action	1. $\psi_1 \in \zeta_k$ 2. For each $\zeta_j : \zeta_j \in Z_\varphi, j < k$, such that the relation $R(\zeta_k, \zeta_j)$ is set, $\psi \in \zeta_j$
$\exists i \Pr^2(i, \theta(\nu_1, \nu_2))$-rule (for predicate letter "=")	Condition	1. $\psi = \exists i \Pr^2(i, \theta(\nu_1, \nu_2))$ 2. $\psi \in \zeta_k$
	Action	If there is no $\zeta_i \in Z_\varphi$, such that $i = \theta(\nu_1, \nu_2)$, then such list is created.
$\forall i \Pr^2(i, \theta(\nu_1, \nu_2))$-rule (for predicate letter "=")	Condition	1. $\psi = \forall i \Pr^2(i, \theta(\nu_1, \nu_2))$ 2. $\psi \in \zeta_k$
	Action	If there is no $\zeta_i \in Z_\varphi$, such that $i = \theta(\nu_1, \nu_2)$, then such list is created.

Table 1 does not contain rules for resolving formulae like $\psi = Fi\psi_1$, where i is a numerical variable, or like $\psi = \forall i\psi_1$. Such formulae can be presented in the form $\forall jFj\psi_1$ with application of the deduction rules R5, R6.

It also should be pointed out that the $\forall iFi$- and $\forall iPi$-rules reflect the transitivity and reflexivity of the relation R between possible worlds at different time points. According to the definition of a model for a modal system (see [12]) the model of the propositional metric temporal calculus *PTC(MT)*, constructed according to the rules from Table 1, is S4-model.

Definition 7.

A set $\{Z_\varphi^1, ..., Z_\varphi^k\}$ of chains constructed according to the rules enlisted in the Table 1, is called a **construction** C_{φ}.

Definition 8.

Chain Z_{φ} is **closed**, if it contains a formulae list ζ such that for some propositional variable p both p and $\neg p$ are in ζ. Construction C_{φ} is **closed** if all chains in it are closed.

Definition 9.

Let φ be a formula of *PTC(MT)*. A **model for** φ will be any chain Z_{φ}, which is not closed.

Definition 10.

Formula φ is **satisfiable** if and only if φ has a model defined over the construction C_{φ}.

Definition 11.

Formula φ is **logically valid** (denoted as $\models$) if and only if $\neg\varphi$ does not have a model defined over the construction $C_{\neg\varphi}$ (in other words, $\neg\varphi$ is **unsatisfiable**).

Metatheorem 1.

If φ is derivable ($\vdash \varphi$), then it is logically valid ($\models \varphi$).

Proof:

Axioms A1-A11 and AMT1-AMT9 (with their mirrors) are logically valid. To prove this one may construct n.n.f. for the negation of each axiom, and applying rules from Table 1 obtain a closed construction.

Every theorem of *PTC(MT)* can be derived from the set of axioms A1-A11 and AMT1-AMT9 (with their mirrors) with the help of the deduction rules R1-R6. As far as the deduction rules preserve the logical validity property, every theorem of *PTC(MT)* is logically valid. Proof is completed.

Metatheorem 2.

PTC(MT) is sound.

Proof: (by the rule of contraries)

Recall that soundness of an axiomatized formal theory is the denial of $\vdash \varphi$ and $\vdash \neg\varphi$ from the same set of axioms and deduction rules.

Let $\vdash \neg\varphi$, and φ is a theorem of *PTC(MT)*. According to the Metatheorem 1, $\neg\varphi$ is logically valid. Logical validity of $\neg\varphi$ requires that $\neg\neg\varphi$ (or simply φ) does not have a model. However, $\neg\neg\varphi$ as the theorem should have a model. Come to the contradiction. Hence, *PTC(MT)* is sound. End of proof.

PTC(MT) completeness proof requires that $\vdash \varphi$ iff $\models \varphi$. This can be achieved by proving of additional statements.

Definition 12.

Let C_φ^n be a construction C_φ on its *n*-th step of creation, $Z_{\varphi,n} \in C_\varphi^n$ be a chain of formulae lists. **Rank *r*** of $Z_{\varphi,n}$ is defined as follows:

$r(\zeta,n)=0$, if there is no $\zeta_i \in Z_{\varphi,n}$ such that $R(\zeta,\zeta_i)$,

$r(\zeta,n)=r(\zeta_i,n)+1$, otherwise.

Intuitively, rank of a chain indicates if formulae lists in the chain are connected.

Definition 13.

Let $\psi_1,\psi_2,...,\psi_s$ be all formulae of the formulae list ζ_j in the chain $Z_{\varphi,n}$ of the construction C_φ^n. Formula $AF(\zeta_j,n)=\psi_1 \wedge \psi_2 \wedge ... \wedge \psi_s$ is called **associated form of the formulae list** ζ_j.

Definition 14.

Characteristic form $CF(\zeta_j,n)$ **of a formulae list** ζ_j in the chain $Z_{\varphi,n}$ of the construction C_φ^n is defined by induction over rank $r(\zeta_j,n)$:

If $r(\zeta_j,n)=0$, then $CF(\zeta_j,n)=AF(\zeta_j,n)$.

If $r(\zeta_j,n)>0$ and there is $\zeta_i \in Z_{\varphi,n}$ such that $R(\zeta_j,\zeta_i)$ and $CF(\zeta_i,n)$ is defined, then

$CF(\zeta_j,n)=AF(\zeta_j,n)\wedge F(i-j)CF(\zeta_i,n)$, if $R(\zeta_j,\zeta_i)$ is directed to the future ($i>j$), and

$CF(\zeta_j,n)=AF(\zeta_j,n)\wedge P(j-i)CF(\zeta_i,n)$, if $R(\zeta_j,\zeta_i)$ is directed to the past ($i<j$).

Definition 15.

Characteristic form $CF(Z_\varphi,n)$ **of a chain** $Z_{\varphi,n}$ of the construction C_φ^n corresponds to the characteristic form of the formulae list ζ_0, $CF(Z_\varphi,n)=CF(\zeta_0,n)$.

Definition 16.

Characteristic form of a construct C_φ^n (denoted as $CF(\varphi,n)$) for the formula φ is

$$CF(\varphi,n)=CF(Z_\varphi^1,n)\vee ... \vee CF(Z_\varphi^k,n).$$

Lemma.

$\vdash CF(\varphi,1)\supset CF(\varphi,n)$

Proof:

It will be enough to show that $\vdash CF(\varphi,m)\supset CF(\varphi,m+1)$ for $1\le m<n$.

According to the definition 14 and to the rules from the Table 1 only the following situations may occur:

$$\vdash CF(Z_\varphi^1,m)\vee ...\vee CF(Z_\varphi^i,m)\vee ...\vee CF(Z_\varphi^k,m)\supset$$
$$\supset CF(Z_\varphi^1,m+1)\vee ...\vee CF(Z_\varphi^i,m+1)\vee ...\vee CF(Z_\varphi^k,m+1)$$

if at the (*m+1*) step of C_φ^n construction was applied one of $\wedge$-rule, $F\nu$-rule, $P\nu$-rule, $\forall iFi$-rule, $\forall iPi$-rule, and

$$\vdash CF(Z_\varphi^1,m)\vee ...\vee CF(Z_\varphi^i,m)\vee ...\vee CF(Z_\varphi^k,m)\supset$$
$$\supset CF(Z_\varphi^1,m+1)\vee ...\vee CF(Z_\varphi^{i1},m+1)\vee CF(Z_\varphi^{i2},m+1)...\vee CF(Z_\varphi^k,m+1)$$

if at the (*m+1*) step of C_φ^n construction was applied one of $\vee$-rule, $\exists iFi$-rule or $\exists iPi$-rule.

As far as $\vdash (p\supset q)\supset((p\vee r)\supset(q\vee r))$ is the theorem of classical propositional calculus, the proof of the Lemma will depend on proofs of the following theorems:

$\vdash CF(Z_\varphi^i,m)\supset CF(Z_\varphi^i,m+1)$ for one of $\wedge$-rule, $F\nu$-rule, $P\nu$-rule, $\forall iFi$- rule, $\forall iPi$-rule, and

$\vdash CF(Z_\varphi^i,m)\supset CF(Z_\varphi^{i1},m+1)\vee CF(Z_\varphi^{i2},m+1)$ for $\vee$-rule, $\exists iFi$-rule or $\exists iPi$-rule.

According to the definition 13, this means that

$\vdash CF(\zeta_0^i,m)\supset CF(\zeta_0^i,m+1)$ for one of $\wedge$-rule, $F\nu$-rule, $P\nu$-rule, $\forall iFi$- rule, $\forall iPi$-rule, and

$\vdash CF(\zeta_0^i,m)\supset CF(\zeta_0^{i1},m+1)\vee CF(\zeta_0^{i2},m+1)$ for $\vee$-rule, $\exists iFi$-rule or $\exists iPi$-rule.

Finally, let $\zeta_j\in Z_\varphi^i$ be a formulae list, which was subjected to the application of rules from the Table 1. Prove that

$\vdash CF(\zeta_j,m)\supset CF(\zeta_j,m+1)$ for one of $\wedge$-rule, $F\nu$-rule, $P\nu$-rule, $\forall iFi$- rule, $\forall iPi$-rule, and

$\vdash CF(\zeta_j,m)\supset CF(\zeta_{j1},m+1)\vee CF(\zeta_{j2},m+1)$ for $\vee$-rule, $\exists iFi$-rule or $\exists iPi$-rule.

Case 1. Let $\psi\in\zeta_j$ be a formula, which was subjected to the application of $\wedge$-rule at the (*m+1*) step, i.e. $\psi=\psi_1\wedge\psi_2$. Associated form of ζ_j at the step *m* is $AF(\zeta_j,m)\equiv\gamma_j\wedge\psi$, where γ_j is a conjunction of all formulae of ζ_j for the step *m* except for ψ.

If rank $r(\zeta_j,m)=0$, then $CF(\zeta_j,m)=AF(\zeta_j,m)=\gamma_j\wedge(\psi_1\wedge\psi_2)$ and following the $\wedge$-rule we obtain $CF(\zeta_j,m+1)=\gamma_j\wedge(\psi_1\wedge\psi_2)\wedge\psi_1\wedge\psi_2$.

If rank $r(\zeta_j,m)>0$, then either

$$CF(\zeta_j,m)=AF(\zeta_j,m)\wedge F(k-j)\,CF(\zeta_k,m)=\gamma_j\wedge(\psi_1\wedge\psi_2)\wedge F(k-j)\,CF(\zeta_k,m)$$

or
$CF(\zeta_j,m) = AF(\zeta_j,m) \wedge P(j-k)\,CF(\zeta_k,m) = \gamma_j \wedge (\psi_1 \wedge \psi_2) \wedge P(j-k)\,CF(\zeta_k,m)$.

At the same time for the (*m+1*) step either
$CF(\zeta_j,m+1) = \gamma_j \wedge (\psi_1 \wedge \psi_2) \wedge \psi_1 \wedge \psi_2 \wedge F(k-j)\,CF(\zeta_k,m+1)$ or
$CF(\zeta_j,m+1) = \gamma_j \wedge (\psi_1 \wedge \psi_2) \wedge \psi_1 \wedge \psi_2 \wedge P(j-k)\,CF(\zeta_k,m+1)$.

As far as at the step (*m+1*) we don't apply any rule for the formulae set ζ_k, then the characteristic form of ζ_k at (*m+1*) step is not changed, i.e. $CF(\zeta_k,m) = CF(\zeta_k,m+1)$.

Additionally, $\vdash p \wedge (q \wedge r) \equiv p \wedge (q \wedge r) \wedge q \wedge r$ - is the theorem from classic propositional calculus.

Thus, $\vdash CF(\zeta_j,m) \supset CF(\zeta_j,m+1)$.

Case 2. Consider only the case when $F\nu$-rule is applied (the situation with $P\nu$-rule will be the same). Let $\psi \in \zeta_j$ be a formula, which was subjected to the application of $F\nu$-rule at the (*m+1*) step, i.e. $\psi = F\nu\psi_1$. Associated form of ζ_j at the step *m* is $AF(\zeta_j,m) \equiv \gamma_j \wedge F\nu\psi_1$, where γ_j is a conjunction of all formulae of ζ_j for the step *m* except for ψ.

If rank $r(\zeta_j,m)=0$, then $CF(\zeta_j,m) = AF(\zeta_j,m) = \gamma_j \wedge F\nu\psi_1$ and following $F\nu$-rule we obtain $CF(\zeta_j,m+1) = (\gamma \wedge F\nu\psi_1) = CF(\zeta_j,m)$, as far as this rule does not affect ζ_j, but it affects ζ_{j+1}: $CF(\zeta_{j+1},m+1) = (\gamma_{j+1} \wedge F(\nu-1)\psi_1)$. γ_{j+1} is a conjunction of all formulae of ζ_{j+1} for the step *m*.

If rank $r(\zeta_j,m)>0$, then either
$CF(\zeta_j,m) = AF(\zeta_j,m) \wedge F(k-j)\,CF(\zeta_k,m) = \gamma_j \wedge F\nu\psi_1 \wedge F(k-j)\,CF(\zeta_k,m)$ or
$CF(\zeta_j,m) = AF(\zeta_j,m) \wedge P(j-k)\,CF(\zeta_k,m) = \gamma_j \wedge F\nu\psi_1 \wedge P(j-k)\,CF(\zeta_k,m)$.

At the same time for the (*m+1*) step either
$CF(\zeta_j,m+1) = \gamma_j \wedge F\nu\psi_1 \wedge F(k-j)\,CF(\zeta_k,m+1)$ or
$CF(\zeta_j,m+1) = \gamma_j \wedge F\nu\psi_1 \wedge P(j-k)\,CF(\zeta_k,m+1)$,

and only for $i=j+1$ $CF(\zeta_{j+1},m+1)$ will be changed,
$CF(\zeta_{j+1},m+1) = \gamma_{j+1} \wedge F(\nu-1)\psi_1 = AF(\zeta_{j+1},m) \wedge F(\nu-1)\psi_1$,

for all other values of *i*, $CF(\zeta_i,m+1)$ remain unchanged.

Thus, basing on T1, T2, deduce that $\vdash CF(\zeta_j,m) \supset CF(\zeta_j,m+1)$

Case 3. Let $\psi \in \zeta_j$ be a formula, which was subjected to the application of $\vee$-rule at the (*m+1*) step, i.e. $\psi = \psi_1 \vee \psi_2$. Associated form of ζ_j at the step *m* is $AF(\zeta_j,m) \equiv \gamma_j \wedge (\psi_1 \vee \psi_2)$, where γ_j is a conjunction of all formulae of ζ_j for the step *m* except for ψ.

If rank $r(\zeta_j,m)=0$, then $CF(\zeta_j,m) = AF(\zeta_j,m) = \gamma_j \wedge (\psi_1 \vee \psi_2)$ and following the $\vee$-rule we obtain $CF(\zeta_j^1, m+1) = \gamma_j \wedge (\psi_1 \vee \psi_2) \wedge \psi_1$ and $CF(\zeta_j^2, m+1) = \gamma_j \wedge (\psi_1 \vee \psi_2) \wedge \psi_2$.

Deduce $\vdash CF(\zeta_j, m) \supset CF(\zeta_j^1, m+1) \vee CF(\zeta_j^2, m+1)$ by the axiom A5.

If rank $r(\zeta_j, m) > 0$, then either

$CF(\zeta_j, m) = AF(\zeta_j, m) \wedge F(k-j)\, CF(\zeta_k, m) = \gamma_j \wedge (\psi_1 \vee \psi_2) \wedge F(k-j)\, CF(\zeta_k, m)$

or

$CF(\zeta_j, m) = AF(\zeta_j, m) \wedge P(j-k)\, CF(\zeta_k, m) = \gamma_j \wedge (\psi_1 \vee \psi_2) \wedge P(j-k)\, CF(\zeta_k, m)$.

At the same time for the (*m+1*) step

$CF(\zeta_j^1, m+1) = \gamma_j \wedge (\psi_1 \vee \psi_2) \wedge \psi_1 \wedge F(k-j)\, CF(\zeta_k, m+1)$ and

$CF(\zeta_j^2, m+1) = \gamma_j \wedge (\psi_1 \vee \psi_2) \wedge \psi_2 \wedge F(k-j)\, CF(\zeta_k, m+1)$, where

$CF(\zeta_k, m) = CF(\zeta_k, m+1)$.

Deduce $\vdash CF(\zeta_j, m) \supset CF(\zeta_j^1, m+1) \vee CF(\zeta_j^2, m+1)$ by the axiom A5.

Case 4. Let $\psi \in \zeta_j$ be a formula, which was subjected to the application of $\forall iFi$-rule at the (*m+1*) step, i.e. $\psi = \forall iFi\, \psi_1$. Associated form of ζ_j at the step *m* is $AF(\zeta_j, m) \equiv \gamma_j \wedge \forall iFi\, \psi_1$, where γ_j is a conjunction of all formulae of ζ_j for the step *m* except for ψ.

If rank $r(\zeta_j, m) = 0$, then $CF(\zeta_j, m) = AF(\zeta_j, m) = \gamma_j \wedge \forall iFi\, \psi_1$ and following the $\forall iFi$-rule obtain $CF(\zeta_j, m+1) = AF(\zeta_j, m) \wedge \psi_1 = \gamma_j \wedge \forall iFi\, \psi_1 \wedge \psi_1$.

Basing on the T1,T2, T9 and rules R3 – R6 deduce that

$\vdash \gamma_j \wedge \forall iFi\, \psi_1 \supset \gamma_j \wedge \forall iFi\, \psi_1 \wedge \psi_1$

If rank $r(\zeta_j, m) > 0$, then either

$CF(\zeta_j, m) = AF(\zeta_j, m) \wedge F(k-j)\, CF(\zeta_k, m) = \gamma_j \wedge \forall iFi\, \psi_1 \wedge F(k-j)\, CF(\zeta_k, m)$ or

$CF(\zeta_j, m) = AF(\zeta_j, m) \wedge P(j-k)\, CF(\zeta_k, m) = \gamma_j \wedge \forall iFi\, \psi_1 \wedge P(j-k)\, CF(\zeta_k, m)$.

For the (*m+1*) step either

$CF(\zeta_j, m+1) = \gamma_j \wedge \forall iFi\, \psi_1 \wedge F(k-j)\, CF(\zeta_k, m+1) \wedge \psi_1$ or

$CF(\zeta_j, m+1) = \gamma_j \wedge \forall iFi\, \psi_1 \wedge P(j-k)\, CF(\zeta_k, m+1) \wedge \psi_1$.

At the same time

$CF(\zeta_k, m+1) = AF(\zeta_k, m+1) \wedge \forall iFi\, \psi_1$.

Check the implication $\vdash CF(\zeta_j, m) \supset CF(\zeta_j, m+1)$:

$\vdash \gamma_j \wedge \forall iFi\, \psi_1 \wedge F(k-j)\, \gamma_k \supset \gamma_j \wedge \forall iFi\, \psi_1 \wedge \psi_1 \wedge F(k-j)(\gamma_k \wedge \forall iFi\, \psi_1)$

Deduce that $\vdash CF(\zeta_j, m) \supset CF(\zeta_j, m+1)$ from T1, T2, T9, T14 – T17 and rules R3 – R6.

Other cases $(\psi = \forall iPi\, \psi_1, \psi = \exists iFi\, \psi_1, \psi = \exists iPi\, \psi_1)$ can be proved analogously.

Metatheorem 3.

If $\models \neg\varphi$, then $\vdash\!\!\!- \neg\varphi$

Proof:

Let φ be in *FnPn*-n.f.

$\models \neg\varphi$ means that the construction C_φ is closed, hence (from the definition of closeness of C_φ) each chain in it is also closed. Closeness of a chain $Z_\varphi \in C_\varphi$ means that there is a formulae list ζ_i such that for some variable p both p and $\neg p$ are in ζ_i. From the propositional axiom set of *PTC(MT)* conclude that $\vdash\!\!\!- \neg AF(\zeta_i, n)$.

$\vdash\!\!\!- \neg CF(Z_\varphi, n)$ is concluded from the theorems T1-T4, hence, following the definition of a characteristic form of a construction C_φ $\vdash\!\!\!- \neg CF(\varphi, n)$.

From the Lemma, $\vdash\!\!\!- \neg CF(\varphi,1)$. Taking into account that $CF(\varphi,1) = \varphi$ (0-rule), deduce $\vdash\!\!\!- \neg\varphi$. End of proof.

Metatheorem 4.

$\vdash\!\!\!- \varphi$ iff $\models \varphi$

Proof:

This metatheorem is concluded from metatheorems 1 and 3. End of proof.

Metatheorem 4 shows completeness of *PTC(MT)*.

4. Conclusions and Future Work

The paper presents a step in the research of the temporal metric logic with respect to the logical properties of a formal theory – completeness and soundness. It is shown that propositional metric temporal calculus is complete and consistent.

Tableau construction rules presented in the paper give a basis for creation a reasoner for first-order metric temporal calculus, which is more convenient for knowledge representation.

The work will be continued in the following directions. First, decidability of *PTC(MT)* should be checked, as far as *PTC(MT)* lacks finite model property. Second, all results obtained for the propositional metric temporal system *PTC(MT)* will be considered for the Description Logics family, which are de facto standard for presentation of ontologies in the Semantic Web.

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